EXERCISES 2.1.4.

(1) Describe convergence with respect to the discrete topology.

(2) Show that metric spaces are Hausdorff. Can you think of a topology not induced by a metric?

(3) Show that a point belongs to the closure of a set if and only if every neighborhood of the point intersects the set non-trivially. Prove that a closed subset A of a compact set B is compact. Hint: include A^c in any open cover of A to get a cover of B.

(4) Show that compact metric spaces are separable. Hint: for each $n \in \mathbb{N}$ cover it by finitely many balls $B_{1/n}(x_i)$, $i \in I_n$, and conclude that the sequence $\{x_i\}$ as $i \in \cup_n I_n$ is dense since some $x_i \in B_\varepsilon(x)$ when $1/n < \varepsilon/2$.

(5) Adapt the proof of the Heine-Borel theorem to show that a metric space is compact if and only if it is complete and *totally bounded*, meaning that for any $r > 0$, the space can be covered by finitely many balls of radius r.

(6) Prove that the closure of a connected subset in a topological space is connected, so connected components are closed in the space, and show that they partition it into maximal connected subsets.

(7) Show that $\langle 0, 1\rangle$ is open in $\mathbb{R}$ but not in $\mathbb{R}^2$.