EXERCISES 3.3.6.

(1) Show that a real convex function on an open interval is continuous on that interval. Hint: Consider a convex function $g\colon \langle a,b\rangle \to \mathbb{R}$ with a,b being possibly $\pm\infty$. Assume $a<s<x<y<t<b$ for real numbers with corresponding points S,X,Y,T on the graph of g. Prove then that Y is above the line through S and X, and below the line through X and T, and conclude that $Y \to X$ as $y \to x$ from the right. Prove similarly that $g(y) \to g(x)$ when $y \to x$ from the left.

(2) Show that if $\|f\|_p = 0$, then $f=0$ *almost everywhere*, meaning that f is only non-zero on a set of measure zero. Hint: Observe that the set of $x \in X$ such that $|f|^p(x) > 0$ is a countable union of sets $A_n = \{x \in X \mid |f|^p(x) > 1/n\}$, and show that

$$\mu(A_n)/n \leq \int |f|^p \, d\mu = 0.$$