EXERCISES 2.3.5.

(1) A *normal* space is a Hausdorff space where pairs of disjoint closed subsets can be separated by disjoint open sets. Show that this happens if and only if for a closed subset E of an open subset A, there is an open set B such that $E \subset B \subset \bar{B} \subset A$.

(2) Show that for a subset E of a metric space (X, d), the formula $f_E(x) = \inf\{d(x, y) \mid y \in E\}$ defines a continuous function $f_E \colon X \to [0, \infty\rangle$ with $f_E(x) = 0$ iff $x \in \bar{E}$. Given disjoint closed sets E, F, show that $f = f_E/(f_E + f_F)$ defines a continuous function $f \colon X \to [0, 1]$ that is 0 on E and 1 on F. Conclude that X is normal.

(3) Show that closed subsets of locally compact Hausdorff spaces are locally compact Hausdorff in the relative topology.

(4) An *n-dimensional topological manifold M* is a Hausdorff space with an *atlas of charts*, that is, an open cover $\{U_i\}$ of M with homeomorphisms $\varphi_i \colon U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$. Prove that any manifold is locally compact. Show that the unit circle $\mathbb{T}$ is a 1-dimensional topological manifold that cannot be *embedded* into $\mathbb{R}$, in that there exists no injective continuous map $\mathbb{T} \to \mathbb{R}$.

(5) Show that the characteristic function of an open (closed) set is lower (upper) semicontinuous. Prove that the sum and supremum (infinum) of lower (upper) semicontinuous functions are lower (upper) semicontinuous.