EXERCISES 2.5.11.

(1) A topological space is *second countable* if it has a countable collection of open subsets generating other open sets as unions of this collection. Show that a subset of a second countable space is second countable in the relative topology. Prove that second countable spaces are separable, and that separable metric spaces are second countable. Conclude that $\mathbb{R}^n$ and any normed space with countable dimension are second countable.

(2) Show that compact Hausdorff spaces are rigid, in that they have no stronger or weaker compact Hausdorff topologies.

(3) The *Tychonoff cube* T is the space $[0, 1]^{\mathbb{N}}$ with product topology. Prove that T is a compact Hausdorff space. Show that the formula $d(x, y) = \sum 2^{-n}|x_n - y_n|$ for $x = \{x_n\}, y = \{y_n\} \in T$ defines a metric on T that induces the topology on T.

(4) Consider the equivalence relation $\sim$ on $\mathbb{R}$ given by $a \sim b$ if $a - b \in \mathbb{Z}$. Show that $\mathbb{R}/\sim$ with the quotient topology is homeomorphic to the unit circle in the relative topology from $\mathbb{C}$.