EXERCISES 1.2.6.

(1) Show that $0v = 0$ for any vector v.

(2) Show that finite dimensional vector spaces are of the form $\mathbb{C}^n$. Show that $\mathbb{C}^n$ can be viewed as a $2n$-dimensional real vector space. Show that n vectors that linearly span an n-dimensional vector space is a basis for it. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{C}^n$ are *equivalent*, meaning that $a\|v\|_1 \leq b\|v\|_\infty \leq c\|v\|_1$ for $a, b, c > 0$ independent of v. Show that completeness of normed spaces is preserved under equivalence.

(3) Show that the arc length between points on the unit circle, defined as $\mathbb{T} \equiv \{z \in \mathbb{C} \,|\, |z| = 1\}$, is a metric different from the Euclidean distance in $\mathbb{C} \cong \mathbb{R}^2$. Show that equality in the Cauchy-Schwarz inequality holds when the two vectors u, v involved are *proportional*, i.e. $u = av$ or $v = au$. Prove the *parallelogram law* $\|u+v\|_2^2 + \|u-v\|_2^2 = 2\|u\|_2^2 + 2\|v\|_2^2$ in any Hilbert space.