EXERCISES 3.1.10.

(1) The *mean value theorem* says that for a real-valued continuous function $f$ on $[a,b]$ that is differentiable on $\langle a,b\rangle$, there is $x \in \langle a,b\rangle$ such that $f(b)-f(a) = f'(x)(b-a)$. To prove this, say $g$ given by $g(y) = (f(b)-f(a))y - (b-a)f(y)$ on $[a,b]$ attains a local maximum or minimum at $x \in \langle a,b\rangle$. Then show that $g'(x) = 0$ by carefully considering the signs of the fraction defining the derivative.

(2) Prove the *fundamental theorem of calculus*, namely, if a real-valued Riemann integrable function $f$ on $[a,b]$ has an anti-derivative $F$, meaning that $F' = f$, then $\int_a^b f(x)\,dx = F(b)-F(a)$. Hint: Picking a Riemann partition $P$ such that $S-s<\varepsilon$, the mean value theorem provides $y_i \in [x_{i-1},x_i]$ such that $F(x_i)-F(x_{i-1}) = f(y_i)(x_i - x_{i-1})$. Show then that $|F(b)-F(a)-\int_a^b f(x)\,dx| < \varepsilon$.

(3) Show that a $\sigma$-algebra is closed under countable intersections, and contains the empty set $\phi$. Prove that the composition of a measurable function with a continuous one is again measurable. Show that any measure $\mu$ is zero on $\phi$, and will always be *increasing*, in that $\mu(A) \leq \mu(B)$ when $A \subset B$.

(4) Show that for $f\colon X \to Y$ between topological spaces, the collection of subsets in $Y$ with Borel measurable inverse images is a $\sigma$-algebra (this holds for any $\sigma$-algebra on $X$). Conclude that if $f$ is measurable, then inverse images of Borel sets are Borel. Show also that compositions of $f$ with Borel maps are Borel maps.

(5) Show that points in $[0,\infty]$ are Borel measurable. Prove that for any set $X$ with a $\sigma$-algebra $M$, a function $f\colon X \to [0,\infty]$ is measurable if and only if $f^{-1}(\langle a,\infty]) \in M$ for all $a>0$.

(6) Show that the pointwise supremum, infinum and limit of a sequence $\{f_n\}$ of measurable functions $X \to [0,\infty]$ are measurable. Hint: Use $(\sup f_n)^{-1}(\langle a,\infty]) = \cup f_n^{-1}(\langle a,\infty])$.

(7) Show that any function $s\colon X \to [0,\infty\rangle$ with finite range is a simple function.

(8) Prove that the Lebesgue integral $\int_A f\,d\mu$ is increasing in both $f$ and $A$, and that it is zero if either $\mu(A) = 0$ or $f = 0$ on $A$.

(9) Prove Fatou's lemma by applying Lebesgue's monotone convergence theorem to the sequence of functions $\inf_{i\geq k} f_i \leq f_k$.